Lecture 12: Blow-Ups

Note Title 10/12/2019

Blow-Up

X scheme

(*) $\mathcal{S}$ = sheaf of graded $\mathcal{O}_X$-algebra, $\mathcal{S}$ quasi-coherent

$\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}^d$

w/
- $\mathcal{S}_0 = \mathcal{O}_X$
- $\mathcal{S}_1$ coherent $\mathcal{O}_X$-module
- $\mathcal{S}$ locally generated by $\mathcal{S}^1$ as $\mathcal{O}_X$-algebra

$U \cong \operatorname{Spec} A \subseteq X$ (affine open) $\rightsquigarrow \operatorname{Proj} \underbrace{\mathcal{S}(U)}_{\text{graded } A\text{-algebra}} \xrightarrow{\pi_U} \operatorname{Spec} A = U$

$U \supseteq U_f \cong \operatorname{Spec} A_f$

$$\begin{tikzcd} \operatorname{Proj} S \arrow[rr] & & \operatorname{Spec} A \\ \operatorname{Proj} S_f \arrow[u] \arrow[rr] & & \operatorname{Spec} A_f = U_f \arrow[u] \end{tikzcd}$$

$\mathcal{S}(U) \cong S$, $\operatorname{Proj} S_f \cong \pi_U^{-1}(U_f)$

$\operatorname{Proj} S = \bigcup_{g \in S^1} D_+(g)$, $D_+(g) \cong \operatorname{Spec} S_{(g)}$

$$\begin{tikzcd} S_{(g)} \arrow[d] & A \arrow[l] \arrow[d] \\ S_{(g)} \otimes_A A_f \cong (S_{(g)})_f & A_f \arrow[l] \end{tikzcd}$$

Therefore, one can glue the local construction to $\operatorname{Proj} \mathcal{S} \longrightarrow X$ together w/ $\mathcal{O}(1)$ on $\operatorname{Proj} \mathcal{S}$

ex. $\mathcal{S} = \mathcal{O}_X[x_0, \cdots, x_n] \rightsquigarrow$

$$\begin{tikzcd} \operatorname{Proj} \mathcal{S} \cong \mathbb{P}^n_X \arrow[r] & X \\ \operatorname{Proj}(\mathcal{S}|_U) \cong \mathbb{P}^n_A \arrow[u, phantom, "\cup"] \arrow[r] & U = \operatorname{Spec} A \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Lemma 1: $\mathcal{S}$ satisfies $(*)$, $\mathcal{L}$ invertible sheaf

$$\mathcal{S}' := \bigoplus_{d \geq 0} \mathcal{S}_d \otimes \mathcal{L}^{\otimes d}$$

Then ① $\mathcal{S}'$ satisfies $(*)$

②
$$\begin{tikzcd} \operatorname{Proj} \mathcal{S} \arrow[rr, "\phi", "\cong"'] \arrow[dr, "\pi"'] & & \operatorname{Proj} \mathcal{S}' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$
w/ $\phi^* \mathcal{O}_P(1) \cong \mathcal{O}_P(1) \otimes \pi^* \mathcal{L}$

pf: $U \subseteq X$ affine open ($\cong \operatorname{Spec} A$) s.t. $\mathcal{L}|_U \xrightarrow[\cong]{\varphi} \mathcal{O}_U \xleftarrow[\cong]{\varphi'} \mathcal{L}|_U$ different trivialization

then $\operatorname{Proj} \mathcal{S}(U) \overset{\varphi}{\cong} \operatorname{Proj} \mathcal{S}'(U) \xleftarrow[\cong]{\varphi'} \operatorname{Proj} \mathcal{S}(U)$

$\varphi = \varphi' \cdot f$, $f \in \Gamma(U, \mathcal{O}_X^*)$

$S = \bigoplus_{d \geq 1} S_d \longrightarrow S$ graded isomorphism

$s_d \longmapsto s_d f^d$

$\mathfrak{p} \longleftarrow\!\!| \ \mathfrak{p} \not\supseteq S_+$

$\longrightarrow \operatorname{Proj} \mathcal{S}(U) \xrightarrow[\text{identity}]{\cong} \operatorname{Proj} \mathcal{S}(U)$

So $\operatorname{Proj} \mathcal{S}(U) \overset{\varphi}{\xrightarrow{\cong}} \operatorname{Proj} \mathcal{S}'(U)$ canonically (does NOT depend on trivializations)

Thus, glue to the global isomorphism

$$\begin{tikzcd} \mathrm{Proj}\,\mathcal{S} \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & & \mathrm{Proj}\,\mathcal{S}' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

Notice that restrict to $\pi^{-1}(U)$ w/ $\mathcal{L}|_U \cong \mathcal{O}_U$

$$\varphi^*\mathcal{O}_{P'}(1) \cong \mathcal{O}_P(1)$$

$S \xrightarrow{\varphi} T$ homo. of graded rings

$\rightsquigarrow \mathrm{Proj}\,T \xrightarrow{\varphi^*} \mathrm{Proj}\,S$ if $\varphi^*(p) \not\supseteq S_+$ ($\Downarrow$ T_+)

$\varphi^*\tilde{M} = \widetilde{M \otimes_S T}$, M: graded S-module

However, one need to check how it glues on $\pi^{-1}(U) \cap \pi^{-1}(U')$

Different trivialization $\rightsquigarrow$ $\mathrm{Aut}(\mathrm{Proj}(\mathcal{S}|_U))$

induced by $\bigoplus_{d\geq 0} \mathcal{S}_d \xrightarrow{\varphi} \bigoplus_{d\geq 0} \mathcal{S}_d$

$s_d \longmapsto s_d \otimes f^d$. $f \in \Gamma(U, \mathcal{O}_U^*)$

The induced action on $\mathcal{O}_P(1)$ is multiplication by f^1.

on $D_+(g)$, $\left(\bigoplus_{d\geq 0} \mathcal{S}_{d+1}\right)_{(g)}$ = deg $\boxed{1}$ part of $\left(\bigoplus_{d\geq 0} \mathcal{S}_{d+1}\right)_g$

$$\frac{s_{d+1} \cdot f^{d+1}}{g^d \cdot f^d}$$

S: graded algebra w/ $S_0 = A$

$\rightsquigarrow \mathrm{Proj}\,S \xrightarrow{\pi} \mathrm{Spec}\,A$

M: A-module

then $\varphi^*\tilde{M} = \widetilde{\underbrace{M \otimes_A S}_{\text{as graded } S\text{-module}}}$

Proposition 1 : $X, \mathcal{S}$ as above

w/ $P = \operatorname{Proj} \mathcal{S} \xrightarrow{\pi} X$ w/ $\mathcal{O}_P(1)$ on P

Then ① π is proper

② If X admits an ample invertible sheaf $\mathcal{L}$

then π is projective w/ $\mathcal{O}_P(1) \otimes \pi^*\mathcal{L}^n$

very ample for $n \gg 0$.

pf: ① It suffices to check

$$\operatorname{Proj}(\mathcal{S}|_{\operatorname{Spec} A}) \cong \pi^{-1}(\operatorname{Spec} A) \xrightarrow[\text{projective}]{} \operatorname{Spec} A \subseteq X \quad \text{is proper}$$

projective $\Downarrow$ proper

② $\mathcal{L}$ ample, then $\mathcal{S} \otimes \mathcal{L}^{\otimes n}$ is base point free

X Noetherian $\quad \mathcal{S}_1 \otimes \mathcal{L}^n$ coherent

generated by finitely many sections $s_0, \cdots, s_N$

$$\oplus \mathcal{O}_X^{N+1} \twoheadrightarrow \mathcal{S}_1 \otimes \mathcal{L}^{\otimes n}$$

$$\rightsquigarrow \quad \mathcal{O}_X[x_0, \cdots, x_N] \longrightarrow \mathcal{S}' := \bigoplus_{d \geq 0} \mathcal{S}_d \otimes (\mathcal{L}^n)^d$$

$$\begin{tikzcd}
\mathbb{P}^N_X \arrow[dr] & \operatorname{Proj} \mathcal{S}' \arrow[l] \arrow[d] \arrow[r, phantom, "\cong"] & \operatorname{Proj} \mathcal{S} \arrow[dl] \\
& X &
\end{tikzcd}$$

$P' = \operatorname{Proj} \mathcal{S}'$, $\quad P = \operatorname{Proj} \mathcal{S}$

Lemma 2

$$\mathcal{O}_{P'}(1) \cong \underbrace{\mathcal{O}_P(1)}_{\text{ample in fibre direction}} \otimes \pi^*\mathcal{L}^n$$

ample in base direction

X: Noetherian, $\mathcal{E}$ locally free $\mathcal{O}_X$-module

$\rightsquigarrow \mathcal{S} = \bigoplus_{d \geq 0} S^d(\mathcal{E})$ satisfies $(*)$

symmetric algebra

$\mathbb{P}(\mathcal{E}) := \operatorname{Proj} \mathcal{S}$

ex. $\mathcal{E}$ free $\mathcal{O}_X$-module of rank $r \Longrightarrow \mathbb{P}(\mathcal{E}) = \mathbb{P}^{r-1}_X$

Proposition 2: $X, \mathcal{E}, \mathbb{P}(\mathcal{E})$ as above

① $\operatorname{rk} \mathcal{E} \geq 2$, $\mathcal{S} \cong \bigoplus_{l \in \mathbb{Z}} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)$ canonically

In particular, $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = 0$, $l < 0$

$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})} \cong \mathcal{O}_X$

$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \mathcal{E}$

② $\pi^* \mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$

pf: ① Restrict to affine open set $U \cong \operatorname{Spec} A \xrightarrow{\subseteq} X$

s.t. $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus k+1}$

$\bigoplus_{d \geq 0} S^d(\mathcal{E})|_U \cong A[x_0, \cdots, x_k] =: S$

$\mathcal{S}|_U = \widetilde{S}$

$\pi_*(\mathcal{O}_{\mathbb{P}}|_{\pi^{-1}(U)}) \cong S$

$\shortparallel$

$\bigoplus_{l \geq 0} \Gamma(\pi^{-1}(U), \mathcal{O}_{\mathbb{P}}(l))$

$\cong$

$\Gamma(U, \pi_* \mathcal{O}_{\mathbb{P}}(l))$

Notice $\Gamma(\mathbb{P}^k_{\mathbb{Z}}, \mathcal{O}(-l)) = 0$, $l < 0$

② affine case $\mathcal{O}_{\mathbb{P}^n_A}(1)$ is generated by global sections

Proposition 3: $X, \mathcal{E}, \mathbb{P}(\mathcal{E})$ as above, $g: Y \to X$

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[dr, "g"'] & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi"] \\ & X \end{tikzcd} \approx \mathcal{L} \text{ invertible sheaf on } Y \text{ w/ } g^*\mathcal{E} \twoheadrightarrow \mathcal{L}$$

easier for the quot scheme construction

pf: $(\Rightarrow)$ $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{O}(1)$ induces $g^*\mathcal{E} \twoheadrightarrow g^*\mathcal{O}(1) = \mathcal{L}$

$(\Leftarrow)$ Restrict to an affine open set $U \cong \operatorname{Spec} A \subseteq X$ w/ $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus(n+1)}$

$$\mathcal{O}_U^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}|_U \iff \exists!\, U \xrightarrow{\varphi} \mathbb{P}^n_A \text{ s.t. } \mathcal{L}|_U \cong \varphi^*\mathcal{O}(1)$$

$\ni x_i$ $\qquad s_i = \varphi^*(x_i)$

these give the information to glue back globally

Blow-up of an Ideal sheaf

X: Noetherian scheme, $\mathcal{J} \subseteq \mathcal{O}_X$ ideal sheaf

$\mathcal{S} := \bigoplus_{d \geq 1} \mathcal{J}^d$ satisfies $(*)$

$\tilde{X} := \operatorname{Proj} \mathcal{S}$ is the blow up of $\mathcal{J}$

or blow up along $Y \leftarrow$ w/ ideal sheaf $\mathcal{J}$

ex. $X = \mathbb{A}^n_k$ $\quad p = (0, \cdots, 0) \in X$

$\mathcal{J} = (x_1, \cdots, x_n)$, $\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{J}^d$ & $\tilde{X} = \operatorname{Proj} \mathcal{S} \xrightarrow{\pi} X$

$\mathcal{J}' = (y_1, \cdots, y_n)$ $\quad y_i$ is just x_i but w/ $\deg(y_i) = 1$

$$\mathcal{J} = k[x_1,\cdots,x_n][y_1,\cdots,y_n] \Big/ (x_i y_j - x_j y_i) \longleftarrow k[x_1,\cdots,x_n][y_1,\cdots,y_n]$$

$$\rightsquigarrow \quad \tilde{X} = \operatorname{Proj} \mathcal{J} \hookrightarrow \operatorname{Proj} k[x_1,\cdots,x_n][y_1,\cdots,y_n] = \mathbb{P}^{n-1}_{\mathbb{A}^n_k} \cong \mathbb{P}^{n-1}_k \times \mathbb{A}^n_k$$

given by equations $x_i y_j = x_j y_i$. $i,j \in \{1,\cdots,n\}$

$\pi^{-1}(x) = pt$, $\pi^{-1}(0) \cong \mathbb{P}^{n-1}_k$

$\#$ 0 : origin

see Theorem 2.

$$f : X \longrightarrow Y. \quad \mathcal{J} \subseteq \mathcal{O}_Y \rightsquigarrow f^*\mathcal{J} := f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

Not necessarily inclusion anymore

ex. $\mathbb{Z} \to \mathbb{Z}$, $x \mapsto nx$ $\quad \overset{\otimes \mathbb{Z}_n}{\rightsquigarrow} \quad \mathbb{Z}_n \to \mathbb{Z}_n$, $x \mapsto 0$

$$f^{-1}\mathcal{J} \cdot \mathcal{O}_X = \operatorname{Im}(f^*\mathcal{J} \to \mathcal{O}_X) \subseteq \mathcal{O}_X$$

Proposition 4: X Noetherian scheme. $\mathcal{J}$ coherent sheaf of ideals

$\tilde{X}$ = blow up of X along $\mathcal{J}$ $\quad \pi : \tilde{X} \to X$

Then ① $\tilde{\mathcal{J}} := \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\tilde{X}}$ is an invertible sheaf

② Y = closed subscheme corresponding to $\mathcal{J}$

$U = X \setminus Y$, $\pi|_U : \pi^{-1}(U) \xrightarrow{\cong} U$

pf: ① Recall $\tilde{X} := \operatorname{Proj} \mathcal{J}$. $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{J}^d$ w/ $\mathcal{O}(1)$

Claim: $\mathcal{O}(1) = \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\tilde{X}}$

$U \subseteq X$ affine open $\quad \mathcal{O}(1)\big|_{\mathrm{Proj}\,\mathcal{S}(U)} = \widetilde{\bigoplus_{d\geq 0} \mathcal{J}(U)^{d+1}} = \widetilde{\mathcal{J}\cdot\mathcal{J}(U)}$

$$\cong \pi^{-1}\mathcal{J}\cdot\mathcal{O}_{\widetilde{X}}\big|_{\mathrm{Proj}\,\mathcal{S}(U)}$$

② $\mathcal{J}|_U \cong \mathcal{O}_U \quad \therefore \quad \mathcal{J}_U = \mathcal{O}_U[x]$

$$\mathrm{Proj}(\mathcal{J}|_U) \cong \mathbb{P}^{1-1}_U \cong U$$

Proposition 5. (Universal Property of Blowing Up)

$X, \mathcal{J}, \pi : \widetilde{X} \longrightarrow X$ as above

$f^{-1}\mathcal{J}\cdot\mathcal{O}_Z$ invertible $\Longrightarrow$

$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!\,\tilde{f}"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

pf: It suffices to prove the case $X = \mathrm{Spec}\,A$ affine
Noetherian

$\widetilde{X} = \mathrm{Proj}\, S, \quad S = \bigoplus_{d\geq 0} I^d, \quad I \lhd A$

$a_0, \cdots, a_n \in I$ generators $\rightsquigarrow A[x_0, \cdots, x_n] \twoheadrightarrow S$ degree preserving

$x_i \longmapsto a_i$

$\rightsquigarrow \widetilde{X} \hookrightarrow \mathbb{P}^n_A$

defined by equations

$F(x_0, \cdots, x_n) \in A[x_0, \cdots, x_n]$

w/ $F(a_0, \cdots, a_n) = 0$

Now $Z \xrightarrow{f} X$ s.t $\overset{\mathcal{L}}{\overset{\parallel}{f^{-1}\mathcal{J}\cdot\mathcal{O}_Z}}$ invertible

$\mathcal{J}$ generated by global sections a_i

$\Rightarrow \mathcal{L}$ generated by global sections $s_i = f^{-1}(a_i)$

$\rightsquigarrow$

$$\begin{tikzcd} Z \arrow[rr, "\exists!\, \tilde{g}"] \arrow[dr] & & \mathbb{P}^n_A \arrow[dl] \\ & \mathrm{Spec} A & \end{tikzcd}$$

s.t $\tilde{g}^*\mathcal{O}(1) = \mathcal{L}$

$\tilde{g}^* x_i = s_i$

If $F(a_0, \dots, a_n) = 0$, $\to F(s_0, \dots, s_n) = 0$

$\Rightarrow F(\tilde{g}^* x_0, \dots, \tilde{g}^* x_n) = 0$

$\Leftrightarrow F \circ \tilde{g}^* = 0$

or $\tilde{g}^*$ factor through $\tilde{X}$

Corollary: $f: Y \to X$ morphism of Noetherian schemes

$\mathcal{J}$: coherent sheaf of ideal on X

$\tilde{X}$: blow up of X along $\mathcal{J}$

$\tilde{Y}$: " Y " $f^{-1}\mathcal{J}\cdot\mathcal{O}_Y$

then

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\exists!\, \tilde{f}"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{X} \arrow[d, "\pi_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

If f is closed immersion

then $\tilde{f}$ is closed immersion

$\tilde{Y}$ called <u>proper transform</u> of Y

pf: • $\pi_Y^{-1}(f^{-1}\mathcal{J}\cdot\mathcal{O}_Y)\cdot\mathcal{O}_{\tilde{Y}} = (f\circ\pi_Y)^{-1}\mathcal{J}\cdot\mathcal{O}_{\tilde{Y}}$ invertible

$\pi_Y^*\mathrm{Im}(f^*\mathcal{J}\to\mathcal{O}_Y) = \mathrm{Im}(\pi_Y^* f^*\mathcal{J}\to\mathcal{O}_{\tilde{Y}})$

• f closed immersion, locally $Y = \mathrm{Spec}(A/\mathfrak{a}) \hookrightarrow \mathrm{Spec}A = X$

$\mathcal{J} = I$. then $A/_{\mathfrak{a}} \oplus \bigoplus_{d>0} I^d \otimes A/_{\mathfrak{a}} \twoheadleftarrow A \oplus \bigoplus_{d>0} I^d$

$\tilde{Y} \hookrightarrow \tilde{X}$

Proposition 6. X: variety$/k$, $0 \neq \mathcal{J} \subseteq \mathcal{O}_X$ ideal sheaf

$\tilde{X} \xrightarrow{\pi} X$: blow-up of X along $\mathcal{J}$

Then ① $\tilde{X}$ variety

② π birational, proper, surjective morphism

③ X quasi-projective$/k$ $\Longrightarrow$ π projective

$\tilde{X}$ quasi-projective

pf: ① X variety$/k$ $\Leftrightarrow$ X integral, separated, of finite type$/k$

I: integral

$S = \bigoplus_{d\geq 0} I^d$ integral

π proper $\Rightarrow$ separated + of finite type

$\Downarrow$

$\tilde{X}$ variety $/k$

② $\underset{\text{closed}}{Supp(\mathcal{O}_X/\mathcal{J})} \subsetneq X$, π induces isomorphism on $\underset{\text{open}}{U \subseteq X}$

$\because X$ integral

thus birational

π proper $\rightarrow$ $\pi(\tilde{X}) \subseteq X$ closed $\longrightarrow$ $\pi(\tilde{X}) = X$

$\pi(\tilde{X}) \supseteq U$

or π surjective

③ X quasi-projective, i.e $\exists$ ample invertible sheaf on X

apply Proposition 1.

Theorem 1: X quasi-projective $/k$. Z variety

$f: Z \longrightarrow X$ birational, projective morphism

Then $\exists$ coherent sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_X$

s.t.

$$\begin{tikzcd} Z \arrow[rr, "\cong"] \arrow[dr, "f"'] & & Bl_{\mathcal{J}} X \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Resolution of singularities (Hironaka '96)

Every variety $/k$, k: characteristic 0.

becomes smooth after blowing up non-singular subvarieties

birational morphism

finitely many times.

This is the fundation of birational geometry.

Theorem 2. X: nonsingular variety $/k$

$\cup\!\!\mid$

Y nonsingular closed subvariety w/ ideal sheaf $\mathcal{J}$

$\pi: \tilde{X} \longrightarrow X$ blow up of $\mathcal{J}$

$\cup\!\!\mid$

$\tilde{Y}$ closed subscheme defined by $\tilde{\mathcal{J}} = \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\tilde{X}}$

Then ① $\tilde{X}$ nonsingular

② $\tilde{Y} \longrightarrow Y$

$\cong$

$\mathbb{P}(\mathcal{J}/\mathcal{J}^2)$

③ $\mathcal{N}_{\tilde{Y}/\tilde{X}} \cong \mathcal{O}_{\mathbb{P}(\mathcal{I}/\mathcal{I}^2)}(-1)$

pf: ② $\tilde{X} = \mathrm{Proj}\left(\bigoplus_{d\geq 0} \mathcal{I}^d\right)$

$Y' \cong \mathrm{Proj}\left(\bigoplus_{d\geq 0} \mathcal{I}^d \otimes \mathcal{O}_X/\mathcal{I}\right) = \mathrm{Proj}\left(\bigoplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}\right)$

Y nonsingular $\Longrightarrow$ $\mathcal{I}/\mathcal{I}^2$ locally free

will prove later

$\mathcal{I}^d/\mathcal{I}^{d+1} \cong S^n(\mathcal{I}/\mathcal{I}^2)$

Thus, $\tilde{Y} \cong \mathrm{Proj}\left(\bigoplus S^d(\mathcal{I}/\mathcal{I}^2)\right) =: \mathbb{P}(\mathcal{I}/\mathcal{I}^2)$

① In particular, $\tilde{Y}$ is locally $Y \times \mathbb{P}^{r-1}$, $r = \mathrm{codim}(Y, X)$

and $\tilde{Y}$ is non-singular.

Proposition 4 $\Longrightarrow$ $\tilde{\mathcal{I}}$ is invertible

$\Longrightarrow$ $\tilde{Y}$ is locally principal

A: Noetherian local ring. $a \in A$ not a zero divisor

A/aA regular $\Longrightarrow$ A regular

$\Longrightarrow$ $\tilde{X}$ is non-singular.

③ Proposition 4 $\Longrightarrow$ $\tilde{\mathcal{I}} = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(1)$

$\mathcal{N}_{\tilde{Y}/\tilde{X}}^{-1} \underset{\text{def}}{\cong} \tilde{\mathcal{I}}/\tilde{\mathcal{I}}^2 \cong \mathcal{O}_{\tilde{X}}(1) \otimes \mathcal{O}_{\tilde{X}}/\tilde{\mathcal{I}} \cong \mathcal{O}_{\tilde{X}}(1)|_{\tilde{Y}} \cong \mathcal{O}_{\tilde{Y}}(1)$